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Inherent Robustness Properties of Quasi-infinite Horizon MPC ⋆

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Abstract: We consider inherent robustness properties of model predictive control (MPC) for continuous-time nonlinear systems with input constraints and terminal constraints. We show that when the linear quadratic control law is chosen as the terminal control law, and the related Lyapunov matrix is chosen as the terminal penalty matrix, MPC with nominal prediction model and bounded disturbances has some degree of inherent robustness. We emphasize that the input constraint sets can be any compact set rather than convex sets, and our results do not rely on the continuity of the optimal cost functional or control law in the interior of the feasible region.

Keywords: Nonlinear MPC, inherent robustness, input constraints, terminal constraints.

1. INTRODUCTION

Since robust MPC methods are much more complex than those developed schemes for nominal MPC, it is of interest to analyze under which conditions nominal MPC can guarantee robustness in the face of specific classes of disturbances. Under the fundamental assumption that the presence of uncertainties and disturbances do not cause any loss of feasibility, the robustness properties of nominal MPC algorithms are proposed (Nicolao et al. (1996); Magni and Sepulchre (1997); Scokaert et al. (1997)). The assumption holds true when the problem formulation does not include state and input constraints and when any terminal constraint used to guarantee nominal stability can be satisfied also in perturbed conditions (Magni and Scattolini (2007)). Findeisen and Allgöwer (2005) show that nominal MPC possesses inherent robustness properties if the optimal cost functional is locally Lipchitz continuous. However, both the resulting MPC control law and value functional associated to the optimization problem defining nominal MPC can be discontinuous (Meadows et al. (1995); Rawlings and Mayne (2009); Fontes (2001)). Grimm et al. (2004) used examples to illustrate that MPC applied to nonlinear systems can produce nominal asymptotic stability without any robustness, when the optimization problem contains state constraints or terminal constraints with short horizons. The works depend on the facts that continuity of the optimal value functional on the interior of the feasibility region is a sufficient condition for robustness, as is continuity of the feedback law on the interior of the feasible region. However, both conditions are hard to verify since in general no explicit expression of the cost functional or the MPC control law can be obtained.

In this paper, we consider the inherent robustness of quasi-infinite horizon MPC of nonlinear systems with input constraints. The optimization problem has a terminal constraint, and the uncertainties are persistent. The analysis does not rely on the assumption of the the continuity of the value functional or of the control law, or the discussion on the continuity of the value functional or of the control law, and are thus both more general and of practical use than previous results. The results show that the degree of robustness depends on the terminal ingredients, such as terminal penalty matrix and terminal set, prediction horizon, the upper bound on disturbances and the logarithmic norm of the system.

The remainder of the paper is organized as follows. The problem is set up in Section 2. Terminal conditions of nominal stability, recursive feasibility of optimization problem and robust stability are given in Section 3. Section 4 concludes the paper with a brief summary.

1.1 Notations and basic definitions

Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}_{[0,\infty)}$ the field of non-negative integer. For a vector $v \in \mathbb{R}^n$, $\|v\|$ the 2-norm and $\|v\|_Q = \sqrt{v^TQv}$ with $Q \in \mathbb{R}^{n\times n}$ and $Q > 0$. Suppose that $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of the eigenvalue of matrix M, $\bar{\sigma}(M)$ the largest singular value of M and $\mu(M)$ is the logarithmic norm of matrix M. $d(\mathcal{X}_1, \mathcal{X}_2)$ is the distance of sets $\mathcal{X}_1$ and $\mathcal{X}_2$. The operation $\oplus$ is the addition of sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, $\mathcal{A} \oplus \mathcal{B} := \{a + b \in \mathbb{R}^n | a \in \mathcal{A}, b \in \mathcal{B}\}$. The

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operation $\ominus$ is the subtraction of sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, where $\mathcal{A} \ominus \mathcal{B} := \{x \in \mathbb{R}^{n_x} | \{x\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$. Denote the set $\mathcal{B}_0 := \{x \in \mathbb{R}^n | x^T x \leq 1\}$, and $\varnothing$ as the set which has no element. Denote $\mathbb{L}^n_{[a,b]}$ as the space of all Lebesgue functions mapping from $[a, b]$ to $\mathbb{R}^n$.

Definition 1. (Rawlings and Mayne (2009))
The Hausforff distance $d(\cdot)$ between two subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbb{R}^n$ is defined by

$$d(\mathcal{X}, \mathcal{Y}) := \max\{\sup_{x \in X} d(x, \mathcal{Y}), \sup_{y \in \mathcal{Y}} d(y, \mathcal{X})\},$$

in which $d(a, \mathcal{M})$ denotes the distance of a point $a \in \mathbb{R}^n$ from a set $\mathcal{M} \subset \mathbb{R}^n$ and is defined by

$$d(a, \mathcal{M}) := \inf_b \{d(a, b) | b \in \mathcal{M}\} \text{ and } d(a, b) := \|a - b\|.$$

Definition 2. (Dahlquist (1959))
The logarithmic norm of a matrix $M \in \mathbb{R}^{n \times n}$ is defined as

$$\mu(M) = \lim_{h \to 0^+} \frac{\| I + hM \| -1}{h},$$

where the symbol $\| \cdot \|$ represents any matrix norm defined in the inner product space with inner product $< x, y >$, I is the compatible dimension identity matrix and $x, y \in \mathbb{R}^n$.

2. PROBLEM SETUP

Consider the continuous-time nonlinear system with exogenous disturbances

$$\dot{x}_R(t) = f(x_R(t), u(t)) + w(t), \tag{1a}$$

$$u(t) \in \mathcal{U}, \tag{1b}$$

$$w(t) \in \mathcal{W}, \tag{1c}$$

where $x_R(t) \in \mathbb{R}^{n_x}$ denotes the system state and $u(t) \in \mathbb{R}^{n_u}$ the control input at time instant t, and $w(t) \in \mathbb{R}^{n_x}$ represents a persistent disturbance. Here, we assume that $\mathcal{W} := \{w \in \mathbb{R}^{n_x} \mid \|w\| \leq \beta\}$, i.e., the norm of the disturbance is bounded. The input constraint set $\mathcal{U}$ is a compact set and contains $0 \in \mathbb{R}^{n_u}$ in its interior.

Remark 2.1. The norm which we adopted in this paper can be any induced norm defined in the inner product space. However, the Hausdorff distance and the vector norm should have the same norm associated with the considered disturbance.

The nominal dynamics of system (1) are defined by

$$\dot{x}(t) = f(x(t), u(t)). \tag{2}$$

The optimization problem in the quasi-infinite horizon MPC is formulated as follows:

Problem 1.

$$\begin{aligned}
&\underset{u(\cdot)}{\text{minimize}}\ J(x, T_p)\\
&\text{subject to}\\
&\dot{x}(t) = f(x(t), u(t)), \quad x(t; x(t), t) = x(t),\\
&u(\tau) \in \mathcal{U} \quad \tau \in [t, t + T_p],\\
&x(t + T_p; x(t), t) \in \mathcal{X}_f,
\end{aligned}$$

where T_p is the prediction horizon, $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite state and input weighting matrices, $J(x(t), T_p) := \int_t^{t+T_p} \|x(s; x(t), t)\|^2_Q + \|u(s; x(t), t)\|^2_R ds + \|x(t + T_p; x(t), t)\|^2_P$ is the cost functional. The positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ is the terminal penalty matrix, and $E\left(x(t + T_p; x(t), t)\right) := \|x(t + T_p; x(t), t)\|^2_P$ is the terminal penalty function. The terminal set $\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\}$ is a level set of the terminal penalty function. The term $u(\cdot; x(t), t)$ denotes the predicted input function related to $x(t)$ and $x(\cdot; x(t), t)$ represents the predicted state trajectory starting from $x(t)$ under the control $u(\cdot; x(t), t)$. For simplicity, denote the optimal value of $J(x(t), T_p)$ as $J^0(x(t), T_p)$. In order to guarantee feasibility and nominal stability, P and $\mathcal{X}_f$ have to satisfy terminal conditions, see (Chen and Allgöwer, 1998) and (Mayne et al., 2000). We will introduce these conditions in the next section.

The goal of this paper is to determine the upper bound on the disturbance, β, as large as possible such that the real system is robustly stable for all $w \in \mathcal{W}$, i.e., the real system under *nominal* MPC controller is robustly stable (inherently robust). Notice here that in Problem 1, the *nominal* system is used as prediction model and no disturbances are taken into account.

Some fundamental assumptions are stated in the following:

Assumption 1. The system state x can be measured instantaneously.

Assumption 2. f is twice continuously differentiable, and $f(0, 0) = 0$. Thus, $0 \in \mathbb{R}^{n_x}$ is an equilibrium of the nominal system.

Assumption 3. The system (2) has a unique solution for any initial condition and any piecewise right-continuous input function $u : [0, T_p] \to \mathcal{U}$.

According to the principle of MPC, the optimization problem will be solved repeatedly, when new measurements are available at the sample instants $t_j = j\delta$, where δ is a sample time and $0 < \delta \leq T_p$, $j \in \mathbb{Z}_{[0,\infty)}$. The applied control is

$$u^*(\tau) := u(\tau; x(t), t), \quad \tau \in [t, t + \delta).$$

3. INHERENT ROBUSTNESS TO PERSISTENT DISTURBANCES

In this section, we discuss the inherent robustness properties of nominal MPC rather than propose a new robust MPC scheme. First, we introduce a common way to construct the terminal set and the terminal penalty function of quasi-infinite horizon MPC, which will play an important role in the analysis of inherent robustness properties.

3.1 Terminal Conditions for Nominal Stability

Lemma 1. (Chen and Allgöwer, 1998) Suppose that the Jacobian linearization of the nominal system at the origin is stabilizable, $K \in \mathbb{R}^{n_u \times n_x}$ is the linear quadratic regulator (LQR) optimal feedback matrix of the linearized system with weighting matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$, where $Q > 0$ and $R > 0$. Then, the Lyapunov equation

$$(A + BK + \kappa I_{n_x})^T P + P(A + BK + \kappa I_{n_x}) = -Q^*$$

admits a unique positive definite and symmetric solution $P \in \mathbb{R}^{n_x \times n_x}$, whenever $\kappa \in [0, \infty)$ satisfies

$$\kappa < -\lambda_{max}(A + BK),$$

where $Q^* = Q + K^T RK$ is positive definite. Furthermore, there exists a constant $\alpha \in (0, \infty)$ specifying a neighborhood $\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\}$ of the origin such that

(1). $Kx \in \mathcal{U}$, for all $x \in \mathcal{X}_f$, i.e., the linear feedback controller respects the input constraints in $\mathcal{X}_f$,
(2). $\mathcal{X}_f$ is positively invariant for the nominal system controlled by the local linear feedback $u = Kx$.

Notice that $u(\tau) = Kx(\tau)$, $\tau \in [t, t+T_p]$ is a feasible solution to Problem 1 provided that $x(t) \in \mathcal{X}_f$.

In the set $\mathcal{X}_f$, $u = Kx$ guarantees that $\frac{dE(x)}{dt} \leq -x^T Q^* x$ along the nominal trajectory, where Q^* is defined in Lemma 1. Then, there exists a positive constant τ such that

$$-x^T Q^* x \leq -\tau x^T P x, \quad \forall x \in \mathcal{X}_f,$$

i.e., the linear control law $u = Kx$ renders the nominal system exponentially stable in $\mathcal{X}_f$ and $\|x(t)\|^2 \leq \|x(t_0)\|^2 e^{-\tau(t-t_0)}$. The decay rate τ can be chosen as $\tau \leq \tau_0 = \lambda_{\min}(Q^*)/\lambda_{\max}(P)$.

The terminal control law $u = Kx$ renders the nominal system exponentially stable in the terminal set $\mathcal{X}_f$ and drives the terminal state to a subset of the terminal set $\mathcal{X}_f$. This will help us understand the behavior of the system dynamics under the terminal control law. Furthermore, we will prove later that the MPC controller has the same robustness properties as the terminal control law.

Lemma 2. If the state $x(t)$ of the nominal system (2) lies in $\mathcal{X}_f$ at time instant t, then there exists an $s \in [t, t+\delta]$ such that the system trajectory under the terminal control law enters into the set

$$\Omega := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \leq e^{-\tau_0 \delta}\alpha\right\}$$

at the time instant s.

Proof: Since $x(t) \in \mathcal{X}_f$, $x(t)^T P x(t) \leq \alpha$. Due to $\frac{dE(x)}{dt} \leq -x^T Q^* x$ and $-x^T Q^* x \leq -\tau_0 x^T P x$, we have $\frac{dE(x)}{dt} \leq -\tau_0 x^T P x$. Therefore, $x(t+\delta)^T P x(t+\delta) \leq e^{-\tau_0\delta}\alpha$. □

Notice that $\Omega \subset \mathcal{X}_f$ since $e^{-\tau_0\delta} < 1$.

In the following, an upper bound on the disturbance, β, is estimated which will preserve the recursive feasibility of Problem 1 if the online optimization problem is feasible at the initial time instant. Then, robust stability is shown based on an auxiliary functional which is continuous in some specified set.

3.2 Robust Recursive Feasibility

In this subsection, we first introduce a general lemma which provides a useful estimate on the deviation of the real system state from the nominal system state. Based on the lemma, we discuss the recursive feasibility of nominal MPC with respect to persistent but bounde disturbances.

Lemma 3. (Dekker and Verwer, 1984) Consider the real system (1) and the nominal system (2), where $f(\cdot,\cdot)$ is a continuously differentiable function. The norm of $\|x_R(t) - x(t)\|$ is, for $t \geq 0$, not larger than the function $\phi(t)$ defined by the scalar differential equation

$$\frac{d\phi(t)}{dt} = v(t)\phi(t) + \|w(t)\|$$
$$\phi(0) \geq \|x_R(0) - x(0)\|$$

for all t and for some fixed input $u(\cdot)$, where $v(t) = \mu\left(\frac{\partial f(x,u)}{\partial x}|_{x(t),u(t)}\right)$ along the trajectory of $x(t)$ and $\mu(\cdot)$ is the logarithmic norm.

It follows from Lemma 3 that the states of the systems (1) and (2) satisfy

$$\|x_R(t) - x(t)\| \leq \|x_R(0) - x(0)\| e^{v_1(t)} + e^{v_1(t)} \int_0^t \|w(s)\| e^{-v_1(s)} ds$$

see (Dekker and Verwer, 1984). There exists a constant v such that $v(t) \leq v$ for all $t \in [0, T_p + \delta]$, since x and u are bounded, and $f(x,u)$ is twice continuously differentiable. Thus, for all $t \in [0, T_p+\delta]$,

$$\|x_R(t) - x(t)\| \leq \|x_R(0) - x(0)\| e^{vt} + \frac{\beta}{v}(e^{vt} - 1), \quad (4)$$

provided that $\|w(t)\| \leq \beta$.

Remark 3.1. Since $\lambda_{\max}(\cdot) \leq \mu(\cdot) \leq \sigma_{\max}(\cdot)$ and $\mu(\cdot)$ can be negative for some systems, (4) provides a less conservative estimate on the deviation than the one based on the Gronwall-Bellman inequality (Khalil, 2002).

The following lemma implies that if the disturbance is small enough, the real system trajectory will stay in a tube along the nominal trajectory during the specified interval $t \in [0, T_p + \delta]$.

Lemma 4. Let $x(t)$ be a solution of the nominal system (2) with $\bar{u}(t) \in \mathcal{U}$ for all $t \in [0, T_p + \delta]$, and $x(0) = x_0$. Suppose that $v(t) \leq v$ for all $t \in [0, T_p + \delta]$. Given $\epsilon > 0$, the trajectory of the real system (1) defined on $[0, T_p + \delta]$, with $x_R(0) = x_0$ and $u(t) = \bar{u}(t)$, lies in the tube

$$S(x_0, \epsilon) := \{(t, x_R) \in [0, T_p + \delta] \times \mathbb{R}^{n_x} \mid \|x_R - x(t)\| \leq \epsilon\}.$$

for all $\beta \in [0, \frac{\epsilon v}{e^{v(T_p+\delta)} - 1}]$, where $\epsilon > 0$ and $x(t)$ is the solution to (2) with the initial state $x(t) = x_0$.

Proof: By continuity of $x(t)$ in t (Chen and Allgöwer, 1998) and the compactness of $[0, T_p + \delta]$, we know that $x(t)$ is bounded on $[0, T_p + \delta]$. Furthermore, the set $S(x_0, \epsilon)$ is a compact set which contains $x(t)$ for all $t \in [0, T_p + \delta]$, due to $w \in \mathcal{W}$.

In the time interval $[0, T_p + \delta]$, because of (4) and $x_R(0) = x(0)$, we have

$$\|x_R(t) - x(t)\| \leq \frac{\beta}{v}(e^{vt} - 1), \quad \forall t \in [0, T_p + \delta].$$

Since $\frac{\beta}{v}(e^{vt} - 1)$ is monotonically increasing in t for fixed v, if β is small enough such that $\frac{\beta}{v}(e^{v(T_p+\delta)} - 1) \leq \epsilon$, then $(t, x_R(t)) \in S(x_0, \epsilon)$ for all $t \in [0, T_p + \delta]$. □

Assumption 4. The upper bound on disturbance is $\beta \leq \beta_0$ with

$$\beta_0 = \frac{(1 - e^{-\tau_0\delta})\alpha}{\lambda_{max}(P)} \frac{v}{(e^{v(T_P+\delta)} - 1)}.$$

Assumption 4 guarantees that the system trajectory of the real system (1) lies in the tube $S(x_0, \epsilon)$ along the nominal system trajectory in the interval $[0, T_p + \delta]$, where $\epsilon := \frac{(1-e^{-\tau_0\delta})\alpha}{\lambda_{max}(P)}$.

Denote the Hausdorff distance of sets $\mathcal{X}_f$ and Ω as $d(\mathcal{X}_f, \Omega)$. Since the sets $\mathcal{X}_f$ and Ω have the same center and shape, we know that

$$d(\mathcal{X}_f, \Omega) = \frac{1}{\lambda_{max}(\alpha^{-1}P)} - \frac{1}{\lambda_{max}(e^{\tau\delta}\alpha^{-1}P)} = \frac{(1 - e^{-\tau\delta})\alpha}{\lambda_{max}(P)}.$$

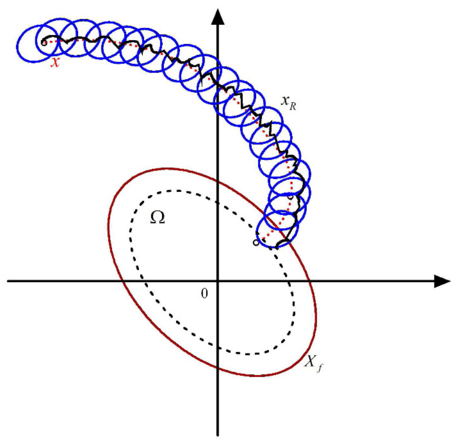

Fig. 1. Tube along the nominal trajectory in $[0, T_p + \delta]$. Dashed line: nominal trajectory, solid line: real trajectory.

Since $\epsilon \le \frac{(1-e^{-\tau_0\delta})\alpha}{\lambda_{max}(P)}$ and $d(\mathcal{X}_f, \Omega) = \frac{(1-e^{-\tau\delta})\alpha}{\lambda_{max}(P)}$, if the terminal state of the nominal system stays in the set Ω, the terminal state of the real system will stay in the terminal set. That is, $x_R(T_p + \delta) \in \mathcal{X}_f$, see Figure 1.

Now we are in a position to state the main result of this subsection, which is about the recursive feasibility of nominal MPC in the presence of disturbances.

Theorem 1. Assume that Problem 1 has a feasible solution at $x(t_0)$, and denote the corresponding predicted nominal control and state as $u(t_0 + \tau; x(t_0), t_0)$ and $x(t_0 + \tau; x(t_0), t_0)$, respectively, $\tau \in [0, T_p]$. Then, $\tilde{u}_{t_0+\delta}(\cdot) \in \mathbb{L}^{n_u}_{[t_0+\delta, t_0+\delta+T_p]}$ with

$$\tilde{u}_{t_0+\delta}(\tau) = \begin{cases} u(\tau; x(t_0), t_0) & \tau \in [t_0 + \delta, t_0 + T_p], \\ Kx(\tau; x(t_0), t_0) & \tau \in (t_0 + T_p, t_0 + \delta + T_p], \end{cases}$$

is a feasible solution to Problem 1 at $x_R(t_0 + \delta)$, where $x_R(t_0 + \delta)$ is a point on the trajectory of the real system starting from $x(t_0)$ under the control $u(\tau; x(t_0), t_0)$, $\tau \in [t_0, t_0 + \delta]$. Furthermore, $(t_0 + \delta, x_R(t_0 + \delta)) \in S(x(t_0), \epsilon)$.

Proof: Following Lemma 2, the control function $\bar{u}_{t_0}(\cdot) \in L^{n_u}_{[t_0, t_0+\delta+T_p]}$ *with*

$$\bar{u}_{t_0}(\tau) := \begin{cases} u(\tau; x(t_0), t_0) & \tau \in [t_0, t_0 + T_p], \\ Kx(\tau; x(t_0), t_0) & \tau \in (t_0 + T_p, t_0 + \delta + T_p], \end{cases}$$

drives the trajectory of the nominal system from $x(t_0)$ *into the set* Ω *in the interval* $[t_0, t_0 + \delta + T_p]$, *i.e.,* $x(t_0 + \delta + T_p; x(t_0), t_0) \in \Omega$, *and the trajectory of the real system with the same control function will lie in the tube* $S(x_0, \epsilon)$ *for all* $t \in [0, t_0 + T_p + \delta]$ *and for all* $w \in \mathcal{W}$.

Thus, $x_R(t_0 + \delta + T_p; x(t_0), t_0) \in \mathcal{X}_f$ *for all* $w \in \mathcal{W}$, *since* $d(\mathcal{X}_f, \Omega) = \frac{(1-e^{-\tau\delta})\alpha}{\lambda_{max}(P)}$ *and* $\epsilon = \frac{(1-e^{-\tau_0\delta})\alpha}{\lambda_{max}(P)}$. *Therefore,* $\tilde{u}_{t_0+\delta}(\cdot)$ *is a feasible solution to Problem 1 at* $x_R(t_0 + \delta)$. □

3.3 Robust Stability

Let $\mathcal{D}$ denote the set of all initial states for which Problem 1 is feasible. Since $u \in \mathcal{U}$ and the prediction horizon T_p is fixed and finite, $\mathcal{D}$ is a bounded set. Theorem 1 guarantees the feasibility of Problem 1 for all time instants, and thus $\mathcal{D}$ is a robust control invariant set for the system (1) under the nominal MPC controller.

In this subsection, we first construct an artificial optimization problem, whose input constraint set is a subset of the original one. We show that the optimal cost functional of the artificial optimization problem, which is an upper bound on the cost functional of the original optimization problem, is continuous in a compact set. Based on this property, we prove that the system trajectory will approach a set around the origin even though there exist small persistent disturbances.

Denote a function $\hat{u}_\tau \in \mathbb{L}^{n_u}_{[0,T_P]}$ with

$$\hat{u}_\tau(s) := \begin{cases} u(s + \tau; x(t_0), t_0) & s \in [0, t_0 + T_p - \tau], \\ Kx(s + \tau; x(t_0), t_0) & s \in (t_0 + T_p - \tau, t_0 + T_p], \end{cases}$$

where $\tau \in [t_0, t_0 + T_p]$ and $\hat{u}_\tau(s)$ is the value of $\hat{u}_\tau$ at time instant s. Notice that $\hat{u}_{t_0}$ is the optimal solution to Problem 1 at $x(t_0)$ and $\hat{u}_\tau$ is a feasible solution to Problem 1 for the initial state $x(\tau)$, $\tau \in [t_0, t_0 + T_p]$, where $x(\tau) := x(\tau; x(t_0), t_0)$ is a point on the predicted trajectory of the nominal system (2) starting from $x(t_0)$ at time instant t_0.

Denote

$$H(x(t_0)) := \{(t, x_R(t; x(t_0), t_0)) \in [t_0, t_0 + T_p] \times \mathbb{R}^{n_x} \mid$$
$$\|x_R(t; x(t_0), t_0) - x(t; x(t_0), t_0)\| \le \frac{\beta}{v}(e^{vt-1}), \forall w \in \mathcal{W}\},$$

which is a reachable set of the system (1) under the control $\hat{u}_{t_0}$. Since $w \in \mathcal{W}$, $H(x(t_0))$ is a compact set and $H(x(t_0)) \subseteq S(x_0, \epsilon)$.

Denote $\mathrm{U}_0 := \{(t, u) \in [t_0, t_0 + T_p] \times \mathbb{L}^{n_u}_{[0,T_P]} \mid u = \hat{u}_t\}$. By continuity of $\hat{u}_\tau$ in x on the predicted trajectory of the nominal system (2) starting from $x(t_0)$ at time instant t_0, and the compactness of $[t_0, t_0 + T_p]$, U_0 is a compact set of trajectories.

For a pair $(t_h, x(t_h)) \in H(x(t_0))$, $t_h \in [t_0, t_0 + T_p]$, define an *artificial* optimization problem

Problem 2.

$$\begin{aligned} &\underset{u}{\text{minimize}}\ \bar{J}(x(t_h), T_p) \\ &\text{subject to} \\ &\dot{x}(t) = f(x(t), u(t)), \quad x(t_h; x(t_h), t_h) = x(t_h), \\ &u \in \mathrm{U}_0, \\ &x(t_h + T_p; x(t_h), t_h) \in \mathcal{X}_f, \end{aligned}$$

where $\bar{J}(x(t_h), T_p) := \int_0^{T_p} \|x(s + t_h; x(t_h), t_h)\|_Q^2 + \|u(s + t_h; x(t_h), t_h)\|_R^2 ds + \|x(t_h + T_p; x(t_h), t_h)\|_P^2$, and $u \in \mathbb{L}^{n_u}_{[0,T_P]}$ is a predicted input trajectory rather than a vector in $\mathbb{R}^{n_u}$. For simplicity, denote the optimal value of $\bar{J}(x, T_p)$ as $\bar{J}^0(x, T_p)$, i.e., $\bar{J}^0(x, T_p) := \min_{u \in \mathrm{U}_0} \bar{J}(x, T_p)$.

Notice that a feasible solution to Problem 2 is also a feasible solution to Problem 1, but an optimal solution to Problem 2 is only a feasible solution to Problem 1, i.e., $J^0(x, T_p) \le \bar{J}^0(x, T_p)$. This stems from the fact that $\hat{u}_\tau(s) \in \mathcal{U}$, for all $s \in [t_0, t_0 + T_p]$.

For $\epsilon \in \left(0, \frac{(1-e^{-\tau_0\delta})\alpha}{\lambda_{max}(P)}\right]$, denote

$$\mathcal{B}(x_s, \epsilon) := \left\{x \in \mathbb{R}^{n_x} \mid (x - x_s)^T (x - x_s) \le \epsilon^2\right\}.$$

Define $U : H(x(t_0)) \to \mathrm{U}_0$ which maps the pair $(t_h, x(t_h)) \in H(x(t_0))$ onto its feasible solutions $u \in \mathrm{U}_0$

of Problem 2. Notice that $U(\cdot)$ is a set-valued function of x in $H(x(t_0))$ since both $\hat{u}_{\tau_1}$ and $\hat{u}_{\tau_2}$ are feasible solutions to Problem 2 at x provided that $x \in \mathcal{B}(x(\tau_1; x(t_0), t_0), \epsilon) \cap \mathcal{B}(x(\tau_2; x(t_0), t_0), \epsilon)$, $\tau_1, \tau_2 \in [t_0, t_0 + T_p]$.

Lemma 5. $\bar{J}^0(x, T_p)$ is continuous at $x(t)$, $(t, x(t)) \in H(x(t_0))$.

Proof: By Proposition 1, the set-valued map $U(\cdot)$ is outer semicontinuous in $H(x(t_0))$ because its graph, $H(x(t_0)) \times \mathrm{U}_0$, is closed. There exists a τ such that for all $(t, x(t)) \in H(x(t_0))$, $x(t) \in \mathcal{B}(x(\tau; x(t_0), t_0), \epsilon)$, since $H(x(t_0)) \subset S(x_0, \epsilon)$. In terms of Theorem 1, the input function $\hat{u}_\tau$ is a feasible solution for all $x' \in \mathcal{B}\big(x(\tau; x(t_0), t_0), \epsilon\big)$. Assume that $\{x_i\}$ is an infinite sequence converging to x', where x_i is the element of the sequence. Then, there exists a constant N such that $x_i \in \mathcal{B}\big(x(\tau; x(t_0), t_0), \epsilon\big)$ for all $i \in \mathbb{Z}_{[N,\infty)}$. Since $U(x_i) \equiv \hat{u}_\tau$ is a feasible solution to Problem 2 at x_i for all $i \in \mathbb{Z}_{[N,\infty)}$, and the sequence $\{U(x_i)\}$ will converge to $\hat{u}_\tau$, it follows from Proposition 2 that $U(\cdot)$ is inner semicontinuous at x. Thus, $U(\cdot)$ is inner semicontinuous at each $(t, x(t)) \in H(x(t_0))$, which further indicates that $U(\cdot)$ is inner semicontinuous in $H(x(t_0))$. Together with the fact that $U(\cdot)$ is outer-semicontinuous, we have that it is continuous in $H(x(t_0))$.

Since $\bar{J}(x, T_p) : H(x(t_0)) \times \mathrm{U}_0 \to [0, \infty)$ is continuous (Rawlings and Mayne, 2009), $U(\cdot)$ is continuous, compact valued and satisfies $U(x(t)) \subset \mathrm{U}_0$ for all $(t, x(t)) \in H(x(t_0))$ where U_0 is compact. Then, $\bar{J}^0(\cdot, T_p)$ is continuous, see Theorem C.28 in (Rawlings and Mayne, 2009). □

Remark 3.2. The optimal cost $J^0(x, T_p)$ is not necessarily continuous in contrast to $\bar{J}^0(x, T_p)$, see (Rawlings and Mayne, 2009).

Since $\bar{J}^0(x, T_p)$ is continuous in the compact set $H(x(t_0))$, it is bounded. Therefore, there exists a parameter $L \geq 0$ such that

$$\bar{J}^0(x_R(t), T_p) - \bar{J}^0(x(t; x(t_0), t_0), T_p) \leq L\|x_R(t) - x(t; x(t_0), t_0)\|, \quad \forall t \in [t_0, t_0 + T_p].$$

Notice that L is only a local Lipschitz constant of $\bar{J}(x, T_p)$ and L is a function of x and β. For fixed x, a smaller β indicates a smaller set $H(x(t_0))$, which further results in a smaller L.

For $s^2 \in \left(0, \frac{\alpha}{\lambda_{\max}(P)}\right]$, denote a set

$$\mathcal{B}_s := \left\{x \in \mathbb{R}^{n_x} \mid x^T x \leq s^2\right\}.$$

Thus, $\mathcal{B}_s \subseteq \mathcal{X}_f$. Denote $L_0 := \max_{x \in \mathcal{D} \ominus \mathcal{B}_s} L(x, \beta_0)$.

The following lemma shows that the state of the real system converges to $\mathcal{B}_s$, if the disturbances are small enough.

Theorem 2. Suppose that

(a). the exogenous disturbance satisfies $\|w(\cdot)\| \leq \beta_s$, where $\beta_s \leq \min\left\{\beta_0, \frac{\rho\delta v s^2 \lambda_{\min}(Q)}{L_0(e^{vt}-1)}\right\}$ and $\rho \in (0, 1)$,
(b). Problem 1 has a feasible solution at the initial time instant t_0.

Then,
(1). the optimization problem is feasible for all $t \geq t_0$,
(2). the system state is robustly asymptotically stable to the set $\mathcal{B}_s$, that is, $\lim_{t\to\infty} d(x_R(t), \mathcal{B}_s) = 0$.

Proof: (1). Recursive feasibility is deduced directly from Theorem 1.

(2). In terms of the definition of $J^0(x, T_p)$, $J^0(0, T_p) = 0$ and $J^0(x, T_p) > 0$ for all $x \neq 0$. Since $J^0(x, T_p) \leq \bar{J}^0(x, T_p)$ for all $x \in H(x(t_0))$,

$$\begin{aligned} &J^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ &\leq \bar{J}^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p). \end{aligned}$$

Since the optimal cost functional is less than any feasible one and (Chen and Allgöwer, 1998)

$$\begin{aligned} &\bar{J}(x(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ &\leq -\int_{t_0}^{t_0+\delta} \big(\|x(s, x(t_0), t_0)\|_Q^2 + \|u(s, x(t_0), t_0)\|_R^2\big) ds, \end{aligned}$$

we have

$$\begin{aligned} &J^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ \leq &\bar{J}^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - \bar{J}(x(t_0 + \delta, x(t_0), t_0), T_p) \\ &- \int_{t_0}^{t_0+\delta} \big(\|x(s, x(t_0), t_0)\|_Q^2 + \|u(s, x(t_0), t_0)\|_R^2\big) ds, \\ \leq &\bar{J}^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - \bar{J}^0(x(t_0 + \delta, x(t_0), t_0), T_p) \\ &- \int_{t_0}^{t_0+\delta} \big(\|x(s, x(t_0), t_0)\|_Q^2 + \|u(s, x(t_0), t_0)\|_R^2\big) ds. \end{aligned}$$

Due to the continuity of $\bar{J}^0(x, T_p)$ in the set $H(x(t_0))$, we have

$$\begin{aligned} &J^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ \leq &L_0\|x_R(t_0 + \delta, x(t_0), t_0) - x(t_0 + \delta, x(t_0), t_0)\| \\ &- \lambda_{\min}(Q) \int_{t_0}^{t_0+\delta} \|x(s, x(t_0), t_0)\|^2 ds \\ &- \lambda_{\min}(R) \int_{t_0}^{t_0+\delta} \|u(s, x(t_0), t_0)\|^2 ds. \end{aligned}$$

In addition, it follows from $(t_0 + \delta, x_R(t_0 + \delta, x(t_0), t_0)) \in H(x(t_0))$ that

$$\|x_R(t_0 + \delta, x(t_0), t_0) - x(t_0 + \delta, x(t_0), t_0)\| \leq \frac{\beta_0(e^{v\delta} - 1)}{v}.$$

Since $\|x(s, x(t_0), t_0)\|^2 \geq s^2$ for all $x(s, x(t_0), t_0) \notin \mathcal{B}_s$, we have

$$\begin{aligned} &J^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ &\leq \frac{L_0\beta_0(e^{v\delta} - 1)}{v} - \lambda_{\min}(Q)\delta s^2, \quad \forall x(s, x(t_0), t_0) \notin \mathcal{B}_s. \end{aligned}$$

Thus,

$$\begin{aligned} &J^0(x_R(t_0 + \delta, x(t_0), t_0), T_p) - J^0(x(t_0), T_p) \\ &< (\rho - 1)\lambda_{\min}(Q)\delta s^2 < 0, \end{aligned}$$

is satisfied if $\|w(t)\| \leq \beta_s$ for all $t \geq t_0$ and $x(s, x(t_0), t_0) \notin \mathcal{B}_s$. Therefore, the system trajectory will enter into the set $\mathcal{B}_s$ in finite time since $J^0(x(t_0), T_p)$ is bounded. Furthermore, the set $\mathcal{B}_s$ is a robust invariant set of the system because the system state will stay in the set with respect to the disturbances $w(\cdot)$, where $\|w(t)\| \leq \beta_s$, for all $t \geq t_0$ □

Corollary 1. Suppose that the disturbance $w(\cdot)$ is decaying, i.e., $\lim_{t\to\infty} \|w(t)\| \to 0$, and $\|w(t)\| \leq \beta_0$, for all $t \geq t_0$. Then, $x = 0$ is asymptotically stable.

Proof: Recursive feasibility is deduced directly from Theorem 1. In case of a decaying disturbance, the disturbance will be arbitrarily small for a large enough time instant. Thus, based on Theorem 2, the system trajectory will approach the set $\mathcal{B}_s$ in finite time under nominal MPC controller, for all $s^2 \in (0, \frac{\alpha}{\lambda_{\max}(P)}]$, and $\lim_{t\to\infty} d(x_R(t), \mathcal{B}_s) = 0$. Thus, together with the results in Theorem 2, $x = 0$ is asymptotically stable. □

Based on the discussion, we conclude that the degree of inherent robustness of quasi-infinite horizon MPC of nonlinear system with respect to bounded disturbances $w(t)$ depends on

(1). the choice of the terminal ingredients,
(2). the upper bound on the disturbance $\|w(t)\|_\infty$,
(3). the prediction horizon T_p,
(4). the logarithmic norm of the considered system.

4. CONCLUSIONS

In this paper, we proved that quasi-infinite horizon MPC of nonlinear systems with input constraints has some robustness. The optimization problem used for the MPC has terminal constraints. Compared with the existing results, the analysis does not rely on the assumptions of convex constraint sets, recursive feasibility and continuity of the cost functional or of the control law. Instead, recursive feasibility was directly proven, and robust stability with respect to persistent disturbances can be shown based on continuity of an auxiliary functional. Logarithmic norm rather than Lipschitz constant was adopted to analyze the metric of robustness, which made the results less conservative.

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REFERENCES

APPENDIX

In what follows, a collection of some well-known properties of set-valued function used in this paper are presented (see for instance, Rawlings and Mayne (2009)).

Definition 3. (Set-valued function)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a function whose value $\phi(x)$ for each x in its domain is a set.

Definition 4. (Outer semicontinuous function)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be outer semicontinuous at x if $\phi(x)$ is closed and if, for every compact set Ω such that $\phi(x) \cap \Omega = \varnothing$, there exists a $\delta > 0$ such that $\phi(x') \cap \Omega = \varnothing$ for all $x' \in x \oplus \delta B_0$. The set-valued function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is outer semicontinuous if it is outer semicontinuous at each $x \in \mathbb{R}^n$.

Definition 5. (Inner semicontinuous function)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be inner semicontinuous at x if, for every open set Ω such that $\phi(x) \cap \Omega \neq \varnothing$, there exists a $\delta > 0$ such that $\phi(x') \cap \Omega \neq \varnothing$ for all $x' \in x \oplus \delta B_0$. The set-valued function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is inner semicontinuous if it is inner semicontinuous at each $x \in \mathbb{R}^n$.

Definition 6. (Continuous function)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is continuous (at x) if it is both outer and inner semicontinuous at x.

Proposition 1. (Outer semicontinuity and closed graph)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is outer semicontinuous in its domain if and only if its graph Z is closed in $\mathbb{R}^n \times \mathbb{R}^m$.

Proposition 2. (Inner semicontinuity)
A set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is inner semicontinuous at x if and only if, for every infinite sequence $\{x_i\}$ converging to x, there exists a $y \in \phi(x)$ and an infinite sequence y_i satisfying $y_i \in \phi(x_i)$ for all i such that y is a limit of $\{y_i\}$.